

Product of Two Consecutive Fibonacci or Lucas Numbers Divisible by their Prime Sum of Indices

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Abstract

We show that the product of two consecutive Fibonacci (respectively Lucas) numbers is divisible by the sum of their indices if this sum is a prime number different from 5 and in the form $(4r + 1)$ (respectively $(4r + 3)$).

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1 Introduction

One of the most interesting divisibility properties of the Fibonacci numbers is that for each prime p , there is a Fibonacci number F_n such that p divides F_n (see, e.g. [5]). More specifically, for $p \neq 5$, p divides either F_{p-1} if $p \equiv \pm 1 \pmod{5}$, or F_{p+1} if $p \equiv \pm 2 \pmod{5}$. For $p = 5$, one has of course $p = F_p$.

2 Theorem

Although already demonstrated differently in [7, 4], a new demonstration of the following theorem is proposed in this paper.

Theorem 1. *If p is prime and $r \in \mathbb{Z}^+$,*

$$p = (4r + 1) \text{ divides the product } F_{2r}F_{2r+1}, \text{ except for } p = 5 \quad (1)$$

$$p = (4r + 3) \text{ divides the product } L_{2r+1}L_{2r+2} \quad (2)$$

Proof. For p prime and $r, s, n, m \in \mathbb{Z}^+$, for odd primes $p = 2s + 1$, one has

$$L_{2s+1} - 1 = L_{2s+1} - L_1 \quad (3)$$

The transformations

$$L_{n+m} - (-1)^m L_{n-m} = 5F_m F_n \quad (4)$$

$$L_{n+m} + (-1)^m L_{n-m} = L_m L_n \quad (5)$$

(relations (17 a, b) in [6] and relations (11) and (23) in [2]) can be used.

(i) First, let s be even, $s = 2r$. Relation (3) yields respectively from (4) and (5), with $m = 2r$ and $n = 2r + 1$,

$$L_{4r+1} - 1 = 5F_{2r} F_{2r+1} \quad (6)$$

$$L_{4r+1} + 1 = L_{2r} L_{2r+1} \quad (7)$$

If $p = 4r + 1 \neq 5$ is prime, then either p divides F_{4r} if $p \equiv \pm 1 \pmod{5} = 29, 41, 61, \dots$, or p divides F_{4r+2} if $p \equiv \pm 2 \pmod{5} = 13, 17, 37, \dots$.

On the other hand, one has (relation (13) in [6])

$$F_{4r} = F_{2r} L_{2r} \quad (8)$$

$$F_{4r+2} = F_{2r+1} L_{2r+1} \quad (9)$$

Let first $p \equiv \pm 1 \pmod{5}$, then p divides F_{4r} and therefore from (8) also either F_{2r} or L_{2r} . But p cannot divide L_{2r} . Let us assume the contrary. Suppose that p divides L_{2r} and also $(L_{4r+1} - 1)$, as $L_p \equiv 1 \pmod{p}$ (see e.g. [1], [3]). It would mean from (7) that p should also divide simultaneously $(L_{4r+1} + 1)$ which makes no sense. Therefore p divides F_{2r} and not L_{2r} , and also $(L_{4r+1} - 1)$. The other case where $p \equiv \pm 2 \pmod{5}$ divides F_{4r+2} is treated similarly.

This means that all primes $p = 4r + 1 \neq 5$ divide the product of two consecutive Fibonacci numbers of indices $2r$ and $2r + 1$. More precisely, if $p \equiv \pm 1 \pmod{5} = 29, 41, 61, \dots$, then p divides F_{2r} ; if $p \equiv \pm 2 \pmod{5} = 13, 17, 37, \dots$, then p divides F_{2r+1} .

(ii) Second, let s be odd, $s = 2r + 1$. Relation (3) yields respectively from (4) and (5), with $m = 2r + 1$ and $n = 2r + 2$,

$$L_{4r+3} + 1 = 5F_{2r+1} F_{2r+2} \quad (10)$$

$$L_{4r+3} - 1 = L_{2r+1} L_{2r+2} \quad (11)$$

If $p = 4r + 3$ is prime, then p divides F_{4r+2} if $p \equiv \pm 1 \pmod{5} = 11, 19, 31, \dots$; or p divides F_{4r+4} if $p \equiv \pm 2 \pmod{5} = 3, 7, 23, 43, \dots$. One has also

$$F_{4r+2} = F_{2r+1} L_{2r+1} \quad (12)$$

$$F_{4r+4} = F_{2r+2} L_{2r+2} \quad (13)$$

Like above, let first $p \equiv \pm 1 \pmod{5}$. Then p divides F_{4r+2} and therefore, from (12), also either F_{2r+1} or L_{2r+1} . But p cannot divide F_{2r+1} . Let us assume the

contrary. Suppose that p divides F_{2r+1} and also $(L_{4r+3} - 1)$. It would mean from (10) that p should also divide simultaneously $(L_{4r+3} + 1)$ which makes no sense. Therefore p divides L_{2r+1} and not F_{2r+1} , and also $(L_{4r+3} - 1)$. The other case where $p \equiv \pm 2 \pmod{5}$ divides F_{4r+4} is also treated similarly.

This means that all primes $p = 4r + 3$ divide the product of two consecutive Lucas numbers of indices $2r + 1$ and $2r + 2$. More precisely, if $p \equiv \pm 1 \pmod{5} = 11, 19, 31, \dots$, then p divides L_{2r+1} ; if $p \equiv \pm 2 \pmod{5} = 3, 7, 23, 43, \dots$, then p divides L_{2r+2} .

On the other hand, for $p = 5$, one has obviously $L_5 = 11 \equiv 1 \pmod{5}$. \square

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References

- [1] P. S. Bruckman, Lucas Pseudoprimes are Odd, *Fibonacci Quarterly* 32, 155-157, 1994.
- [2] R. A. Dunlap, *The Golden Ratio and Fibonacci Numbers*, World Scientific Press, 1997.
- [3] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, p. 410, 2001.
- [4] J. Seibert, Fibonacci and Lucas Products Modulo A Prime, *Solution Problem B-1037*, *Fibonacci Quarterly*, Vol. 46-47, p. 88, 2008-2009
- [5] M. R. Schroeder, *Number Theory in Science and Communication*, 2nd edition, Springer-Verlag, 1986, 72-73.
- [6] S. Vajda, *Fibonacci and Lucas Numbers, and The Golden Section: Theory and Applications*, Halsted Press, 1989.
- [7] H. C. Williams, *Edouard Lucas and primality testing*, *Canadian Math. Soc. Monographs* 22, Wiley, New York, 1998.